Rational solutions of integrable nonlinear wave models

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International Workshop on Integrable Systems
Mathematical Analysis and Scientific Computing
Taipei, October 17-21, 2015

HISTORY 1

LINEAR differential equations with constant coefficients
do not have rational solutions

(1977–78) Adler, Airault, McKean, Moser, Ablowitz, Newell, Satsuma
Korteweg-deVries equation $u_t + u_{xxx} - 6uu_x = 0$

$$u_n(x,t) = -2\partial_x^2 \log(P_n(x,t)) \ , \quad n \geq 0$$

Adler-Moser polynomials : $P_0 = 1$, $P_1 = x$, $P_2 = x^3 + 12t$, ...

$$u_0 = 0 \, , \; u_1 = \frac{2}{x^2} \, , \; u_2 = 6x\frac{x^3 - 24t}{(x^3 + 12t)^2} \, , \; ...$$

Boussinesq equation $u_{tt} \pm u_{xxxx} + (u^2)_{xx} = 0$
motion of poles as many-body system

HISTORY 2

connection to Painleve' II and IV : (1959–1965) Yablonskii–Vorob'ev polynomials, (1999) Noumi, Yamada (generalized Hermite polynomials and generalized Okamoto polynomials)

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defocusing Nonlinear Schroedinger equation $\quad iu_t + u_{xx} - 2|u|^2 u = 0$
(1985) Nakamura, Hirota, (1996) Hone, (2006) Clarkson

$$u_n = \frac{g_n}{f_n} \ , \quad n \geq 0$$

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focusing Nonlinear Schroedinger equation $\quad iu_t + u_{xx} + 2|u|^2 u = 0$
(1983) Peregrine, (2010) Clarkson, Matveev

$$u_n = \frac{G_n}{F_n} \, e^{2it} \ , \quad n \geq 0$$

$$G_0 = 1 \, , \; F_0 = 1 \, , \; G_1 = 4x^2 + 16t^2 - 4it - 3 \, , \; F_1 = 4x^2 + 16t^2 + 1 \, , \; \ldots$$

PEREGRINE LUMP

rational soliton as ratio of polynomials of degree 2

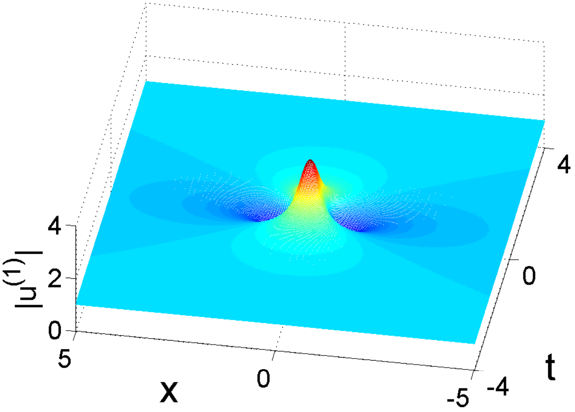

Figure: background amplitude=1 , peak amplitude = 3

HISTORY 3

" The finite density boundary conditions have meaningful applications only when $\chi > 0$, hence we shall confine ourselves to this case. "
L. Faddeev and L. Takhtajan *Hamiltonians Methods in the Theory of Solitons*, Springer (1986)

recent extensions to other integrable models such as:

- vector nonlinear Schroedinger equations
- Hirota equation and coupled Hirota equations
- three wave resonant interaction model
- Massive Thirring Model
- discrete NLS equation
- several others ....

GENERAL OBSERVATIONS

- making a limit :

$$M(z) = \sum_{j=1}^{N+1} \gamma_j e^{k_j z} \to e^{k_c z} P_{(N)}(z) = e^{k_c z} \sum_{j=0}^{N} c_j z^j \ , \ k_j \to k_c$$

- computing the critical value k_c

Example : KdV for Adler-Moser polynomials , $k_c = 0$

Example : NLS for Peregrine and higher order , $k_c = \pm i$

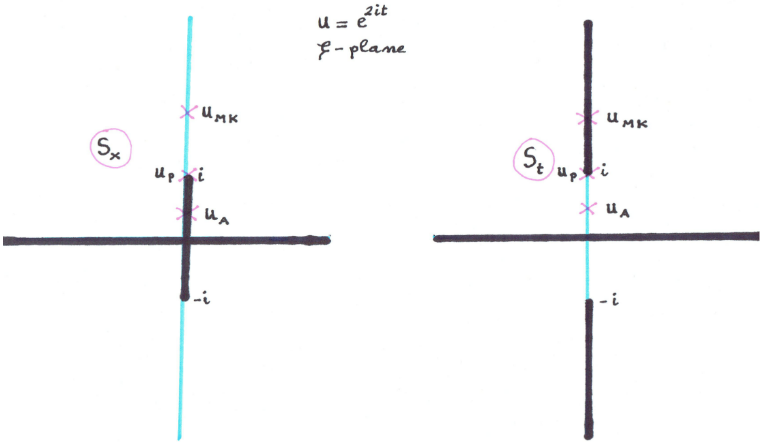

Figure: S_x = x-part continuum spectrum / S_t = t-part continuum spectrum preliminary note on Jordan forms : $M = TM^{(J)}T^{-1}$

$$M^{(J)} = \{n_j x n_j \ \text{blocks}\} = \{m_j \mathfrak{I}_{n_j x n_j} + \mu_j \mathfrak{I}_{n_j x n_j}\}$$

$\mathfrak{I}_{n_j x n_j}$ is the $n_j x n_j$ unit matrix and $\mathfrak{I}_{n_j x n_j} = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$

n_j is the algebraic multiplicity of the eigenvalue m_j and $\mathfrak{I}_{n_j x n_j}^{n_j} = 0$
if $N^n \neq 0$ and $N^{n+1} = 0$ then $e^{zN} = P_n(z)$

$$e^{zM} = T\{e^{zm_j} P_{n_j - 1}(z)\}T^{-1}$$

necessary condition for $\mu_j \neq 0$ is $n_j > 1$

example : NLS equation

$$u_t = i[u_{xx} - 2s\,|u|^2] \ , \ \ \Psi_x = X\Psi \ \ , \ \ \Psi_t = T\Psi \ \ , \ s = \pm 1$$

$$u_0(x,t) = ae^{-isa^2 t} \ , \ \Psi_0(x,t,k) = G(x,t)e^{i(\Lambda(k)x - \Omega(k)t)}$$

DEFINITION : *k_c is a critical value of k if $\Lambda(k_c)$ is similar to a Jordan form Λ_J* :

$$\Lambda(k_c) = T\,\Lambda_J\,T^{-1}$$

$$\Lambda(k) = \begin{pmatrix} k & -isa \\ -ia & -k \end{pmatrix} \ , \ \ \lambda_1 = \sqrt{k^2 - sa^2} \ , \ \lambda_2 = -\sqrt{k^2 - sa^2}$$

$$\text{for } s = 1 \ , \ k_c = \pm a \ , \ \text{for } s = -1 \ , \ k_c = \pm ia \ , \ \Lambda^2(k_c) = 0$$

$$e^{i\Lambda(k_c)x} = 1 + i\Lambda(k_c)x$$

$$\begin{cases} u_t^{(1)} = i[u_{xx}^{(1)} - 2(s_1 \left|u^{(1)}\right|^2 + s_2 \left|u^{(2)}\right|^2)u^{(1)}] \\ u_t^{(2)} = i[u_{xx}^{(2)} - 2(s_1 \left|u^{(1)}\right|^2 + s_2 \left|u^{(2)}\right|^2)u^{(2)}] \end{cases}$$

$$\Psi_x = X\Psi \quad , \quad \Psi_t = T\Psi$$

$$X(x,t,k) = ik\sigma + Q(x,t) \quad , \quad T = 2ik^2\sigma + 2kQ + i\sigma(Q^2 - Q_x)$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \; Q = \begin{pmatrix} 0 & s_1 u^{(1)*} & s_2 u^{(2)*} \\ u^{(1)} & 0 & 0 \\ u^{(2)} & 0 & 0 \end{pmatrix}$$

study case : vector NLS equation 2)

$$\Psi(x,t,k) = \left[\mathbf{1} + \left(\frac{\chi-\chi^*}{k-\chi}\right) P(x,t)\right] \Psi_0(x,t,k)$$

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} u_0^{(1)}(x,t) \\ u_0^{(2)}(x,t) \end{pmatrix} + \frac{2i(\chi-\chi^*)\zeta^*}{|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$P(x,t) = \frac{ZZ^\dagger}{|\zeta|^2 - s_1|z_1|^2 - s_2|z_2|^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -s_1 & 0 \\ 0 & 0 & -s_2 \end{pmatrix}$$

$$Z(x,t) = \begin{pmatrix} \zeta(x,t) \\ z_1(x,t) \\ z_2(x,t) \end{pmatrix} = \Psi_0(x,t,\chi^*) Z_0$$

$$\begin{pmatrix} u_0^{(1)}(x,t) \\ u_0^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} a_1 e^{i(qx-\nu t)} \\ a_2 e^{-i(qx+\nu t)} \end{pmatrix} \;,\; \nu = q^2 + 2(s_1 a_1^2 + s_2 a_2^2) \;,\; a_j > 0$$

$$\Psi_0(x,t,k) = G(x,t) e^{i(\Lambda(k)x - \Omega(k)t)} \;,\; [\Lambda(k)\,,\,\Omega(k)] = 0$$

$$Z(x,t) = G(x,t) e^{i(\Lambda(\chi^*)x - \Omega(\chi^*)t)} Z_0$$

$$\Lambda(k) = \begin{pmatrix} k & -is_1 a_1 & -is_2 a_2 \\ -ia_1 & -k-q & 0 \\ -ia_2 & 0 & -k+q \end{pmatrix}$$

$$P_\Lambda(\lambda) = \det[\lambda - \Lambda(k)] = \lambda^3 + A_2(k)\lambda^2 + A_1(k)\lambda + A_0(k)$$

$$\Delta(k) = \text{discriminant of } P_\Lambda(\lambda) = k^4 + D_3 k^3 + D_2 k^2 + D_1 k + D_0$$

classification of rational solutions by computing :

1. the critical value k_c

$$\Delta(k_c) = 0 \ , \ \ k_c \neq k_c^*$$

2. the similarity matrix T, the Jordan form Λ_J and the matrix $\widehat{\Omega}$

$$\Lambda(k_c) = T\,\Lambda_J\,T^{-1} \ , \ \Omega(k_c) = T\,\widehat{\Omega}\,T^{-1} \ , \ [\Lambda_J\,,\,\widehat{\Omega}] = 0$$

3. the vector

$$Z(x,t) = G(x,t)\,Te^{i(\Lambda_J x - \widehat{\Omega} t)} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

Case $[\lambda_1 = \lambda_2 = \lambda_3]$

$$q \neq 0\,, \quad a_1 = a_2 = 2q\,, \quad s_1 = s_2 = -1\,, \quad k_c = \pm i\frac{\sqrt{27}}{2}q$$

$$\Lambda_J = \begin{pmatrix} \lambda_1 & \mu_1 & 0 \\ 0 & \lambda_1 & \mu_1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \;, \quad \widehat{\Omega} = \begin{pmatrix} \omega_1 & \rho_1 & \rho_2 \\ 0 & \omega_1 & \rho_1 \\ 0 & 0 & \omega_1 \end{pmatrix}$$

$$\lambda_1 = -\frac{k_c}{3}\;,\; \mu_1 = 2iq\;,\; \omega_1 = \frac{11}{2}q^2\;,\; \rho_1 = 4\sqrt{3}q^2\;,\; \rho_2 = 4q^2$$

$$T = \begin{pmatrix} \theta & 0 & -i \\ 1 & \theta^* & i\sqrt{3} \\ i\theta^* & i & 0 \end{pmatrix} \;, \quad \theta = \frac{1}{2}(-\sqrt{3} + i)$$

CLASSIFICATION - 2

1. $\gamma_3 = 0$

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{i(qx-\nu t)} & 0 \\ 0 & e^{-i(qx+\nu t)} \end{pmatrix} \frac{1}{P_2} \begin{pmatrix} P_2^{(1)} \\ P_2^{(2)} \end{pmatrix}$$

2. $\gamma_2 = 0$

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = \begin{pmatrix} e^{i(qx-\nu t)} & 0 \\ 0 & e^{-i(qx+\nu t)} \end{pmatrix} \frac{1}{P_4} \begin{pmatrix} P_4^{(1)} \\ P_4^{(2)} \end{pmatrix}$$

VNLS rational solutions 1 ($\lambda_1 = \lambda_2 = \lambda_3$)

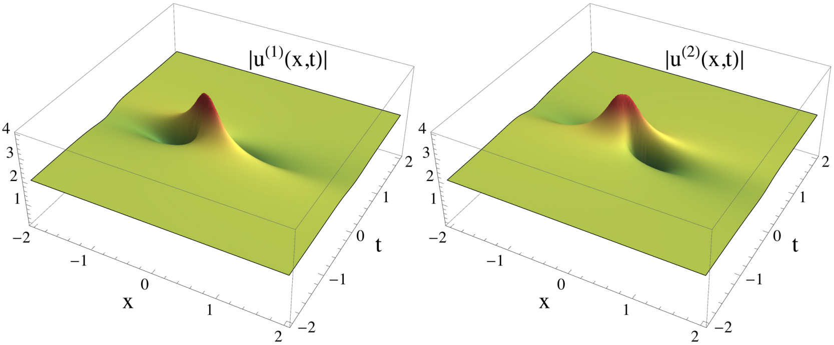

Figure: $k_c = i\frac{\sqrt{27}}{2}$, $s_1 = s_2 = -1$, $q = 1$, $a_1 = a_2 = 2$; $\gamma_2 = 1$, $\gamma_1 = \gamma_3 = 0$.

VNLS rational solutions 2 ($\lambda_1 = \lambda_2 = \lambda_3$)

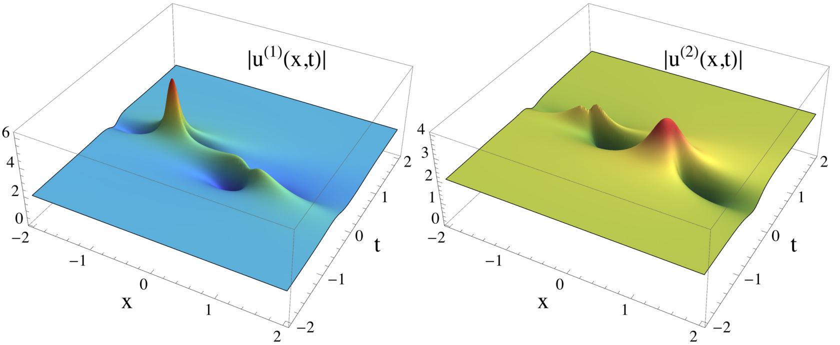

Figure: $k_c = i\frac{\sqrt{27}}{2}, s_1 = s_2 = -1, q = 1, a_1 = a_2 = 2, \gamma_1 = i, \gamma_2 = 0, \gamma_3 = 1.$

CLASSIFICATION - 3

$$\text{Case} \quad [\lambda_1 = \lambda_2 \neq \lambda_3]$$

$$\Lambda_J = \begin{pmatrix} \lambda_1 & \mu & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \; , \; \widehat{\Omega} = \begin{pmatrix} \omega_1 & \rho & 0 \\ 0 & \omega_1 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}$$

1. $q = 0 \;,\; s_1 = s_2 = -1$ explicit analytical
2. $q \neq 0 \;,\; s_1 = s_2 \;,\; a_1 = a_2$ explicit analytical
3. $q \neq 0 \;,\; a_1 \neq a_2$ numerical

VNLS rational solutions 3 ($\lambda_1 = \lambda_2 \neq \lambda_3$)

$q = 0$, $s_1 = s_2 = -1$ vector Peregrine solution

$$\begin{pmatrix} u^{(1)}(x,t) \\ u^{(2)}(x,t) \end{pmatrix} = e^{2i\omega t}\left[\frac{L}{B}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \frac{M}{B}\begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}\right]$$

$$L = P_2 + |f|^2 e^{2px} \ , \ M = 4fe^{px+i\omega t}P_1 \ , \ B = \hat{P}_2 + |f|^2 e^{2px}$$

$$k_c = \pm ip \ , \ p = \sqrt{a_1^2 + a_2^2} \ , \ \omega = a_1^2 + a_2^2$$

$$\lambda_1 = \lambda_2 = 0 \, , \, \lambda_3 = -ip \, , \, \mu = -ip \, , \, \omega_1 = \omega_2 = p^2 \, , \, \omega_3 = 0 \, , \, \rho = -2p^2$$

$$T = \begin{pmatrix} -p & p & 0 \\ a_1 & 0 & a_2 \\ a_2 & 0 & -a_1 \end{pmatrix}$$

VNLS rational solutions 4 ($\lambda_1 = \lambda_2 \neq \lambda_3$)

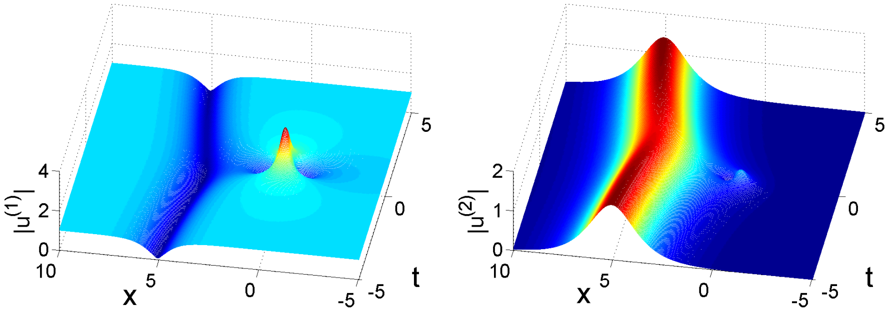

Figure: $k_c = i, q = 0, a_1 = 1, a_2 = 0, s_1 = s_2 = -1, f = 0.1,$

VNLS rational solutions 5 ($\lambda_1 = \lambda_2 \neq \lambda_3$)

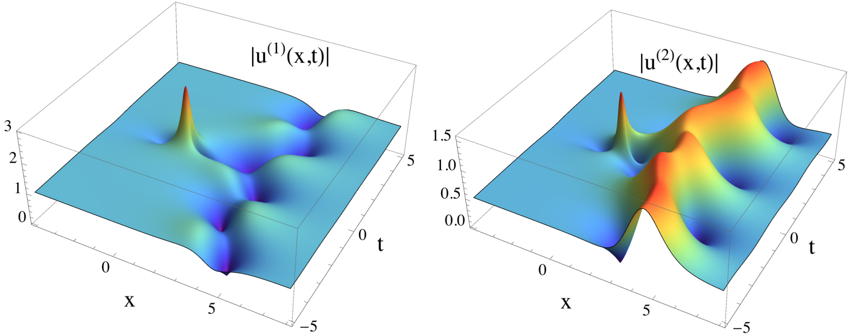

Figure: $k_c = i\frac{\sqrt{5}}{2}, , q = 0, a_1 = 1, a_2 = 0.5, s_1 = s_2 = -1, f = 0.1i$

VNLS rational solutions 6 ($\lambda_1 = \lambda_2 \neq \lambda_3$)

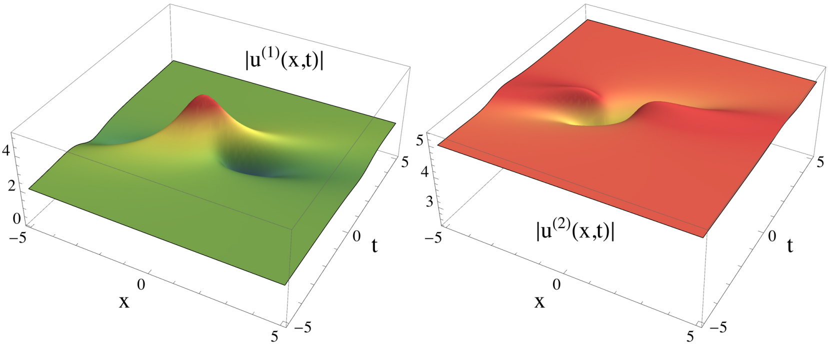

Figure:
$k_c = 4.876 + 5.343i, q = 1, a_1 = 2, a_2 = 5, s_1 = s_2 = -1, \gamma_2 = 1, \gamma_1 = \gamma_3 = 0$

VNLS rational solutions 7 ($\lambda_1 = \lambda_2 \neq \lambda_3$)

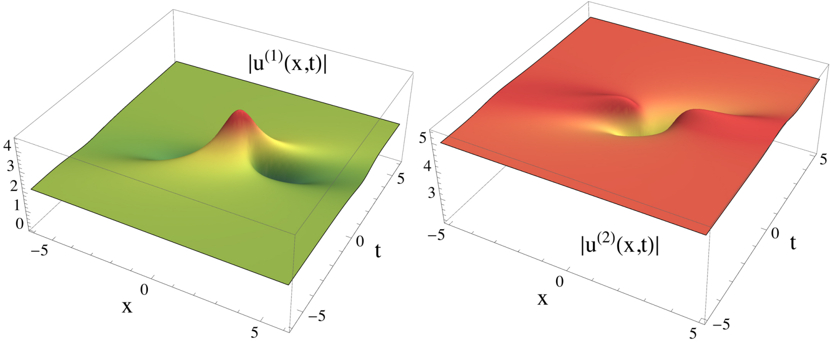

Figure:
$k_c = -5.600 + 4.655i, q = 1, a_1 = 2, a_2 = 5, s_1 = s_2 = 1, \gamma_2 = 1, \gamma_1 = \gamma_3 = 0$

VNLS rational solutions 8 ($\lambda_1 = \lambda_2 \neq \lambda_3$)

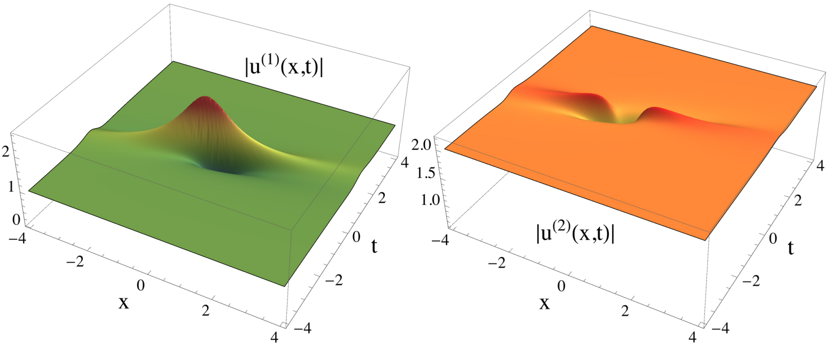

Figure: $k_c = -1.242 + 0.636i, q = 1, a_1 = 2, a_2 = 2, s_1 = -1, s_2 = 1, \gamma_2 = 1, \gamma_1 = \gamma_3 = 0$

other integrable equations

3 wave resonant interaction equations :

$$\begin{cases} E_{1t} + V_1 E_{1x} & = & E_2^* E_3^* \\ E_{2t} + V_2 E_{2x} & = & -E_1^* E_3^* \\ E_{3t} + V_3 E_{3x} & = & E_1^* E_2^* \end{cases}$$

Massive Thirring Model equations :

$$\begin{cases} iU_\xi - \nu V & = & \frac{1}{\nu}|V|^2 U \\ iV_\eta - \nu U & = & \frac{1}{\nu}|U|^2 V \end{cases}$$

$$\partial_\xi = \partial_t + c\partial_x \quad , \quad \partial_\eta = \partial_t - c\partial_x$$

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